Modal Logic and Kripke Models

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- The purpose of a logic is to characterise a notion of logical consequence, to know the difference between valid and invalid arguments.
- One expects a logic to have:
  - A language (the syntax: alphabet, well-formed formulas).
  - A proof system (a set of axioms and rules of inference and the derivability relation ($\vdash$)).
  - A semantics (the notion of truth, and the entailment relation ($\models$)).

Language of Modal Logic

Definition (Basic Syntax)

A (well-formed) formula A is inductively defined by the following grammar:

$$A, B ::= \quad \Sigma \mid \top \mid \bot \mid \neg A \mid (A \wedge B) \mid (A \vee B) \mid (A \to B) \mid \Box A \mid \Diamond A \quad \text{(formulas)}$$

where A, and B are metavariables for formulas, and Σ stands for any propositional variable, usually denoted as $p, q, r, \cdots$. We call $\Box$ and $\Diamond$ the modal operators.

- $\top$ and $\bot$ are logical constants for *always true* and *always false*.
- We could have defined an equivalent language with only $\neg$ and $\to$, and define $\wedge$, and $\vee$ in terms of these as in propositional logic.
- The formula $\Box A$ is read as "it is *necessarily* true that A".
- $\Diamond A$ can be read as "it is *possibly* true that A".

$\Box$ and $\Diamond$

- Many notions in natural language come in dual pairs:
  - "always" and "sometimes",
  - "necessarily" and "possibly",
  - "obligation" and "permission",
  - "already" and "not yet", etcetera.
- The following two principles are intuitively valid:
  - $\Diamond A \leftrightarrow \neg\Box\neg A$
  - $\Box A \leftrightarrow \neg\Diamond\neg A$

  The same pattern is found in First-order logic:
  - $\exists x A(x) \leftrightarrow \neg\forall x \neg A(x)$
  - $\forall x A(x) \leftrightarrow \neg\exists x \neg A(x)$
- So take either modality as *primitive*. Here, let's say $\Diamond := \neg\Box\neg A$.

Family of Modalities

- Modal logic extends classical logic by incorporating operators that express *modality*.
- Modalities are expressions/ways to qualify the truth of a judgment.

Modal Logic	$\Box$	it is *necessary* that
	$\Diamond$	it is *possible* that
Deontic Logic	$\mathcal{O}$	it is *obligatory* that
	$\mathcal{F}$	it is *forbidden* that
	$\mathcal{P}$	it is *permitted* that
Temporal Logic	$\mathcal{W}$	it will be the case that
Linear Logic	$\mathcal{N}$	next time it will be the case that
Epistemic Logic	$\mathcal{K}$	I *know* that
Doxastic Logic	$\mathcal{B}$	I *believe* that

- Example. "eventually the program will terminate", or
- Example. "it is always the case that the program it is never deadlocked".

Proof System

- The logical system for a language is a set of axioms and rules of inference designed to prove exactly the valid arguments in the language.
- All the modal logics part from the **K** system:

K system: normal modal logic

- Propositional tautologies such as:
  - $A \to A$
  - $A \to (B \to A)$
  - $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$
  - $(\neg A \to \neg B) \to (B \to A)$
  - Modus Ponens: $(A \to B) \to ((A \to B) \to A)$
- (G) Necessitation: If A is a theorem, then $\Box A$ is a theorem.
- (K) Distribution: $\Box(A \to B) \to (\Box A \to \Box B)$

Axioms of Modal Logics

On top of **K** system (normal modal logic), we can add the following axioms to obtain different modal logics:

Axiom	**Formula**
D	$\Box A \to \Diamond A$
T	$\Box A \to A$
B	$A \to \Diamond\Box A$
4	$\Box A \to \Box\Box A$
5	$\Diamond A \to \Box\Diamond A$

Modal Logic	**Axioms**
KT	**K** + **T**
KD	**K** + **D**
KTB	**K** + **T** + **B**
K4	**K** + **4**
S4	**KT** + **4**
KTB4	**K** + **T** + **B** + **4**
KTB5	**K** + **T** + **B** + **5**
S5	**KTB4**, **KTB5**

So, for example, we have that:

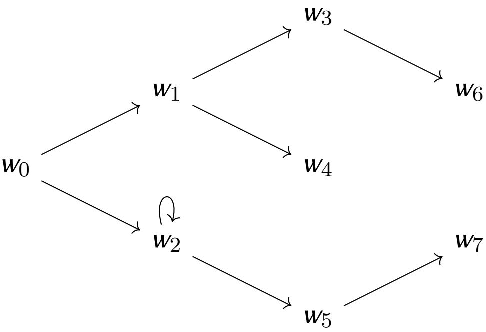
- In **S4**, $\Box^n A \leftrightarrow \Box A$ for all $n \in \mathbb{N}$.
- In **S5**, $(\Box|\Diamond)^n\Box A \leftrightarrow \Box A$ for all $n \in \mathbb{N}$, and $(\Box|\Diamond)^n\Diamond A \leftrightarrow \Diamond A$ for all $n \in \mathbb{N}$.

Formal semantics for modal logics

- The *semantics* for a logic, provides a definition of *entailment* and *validity* by characterising the *truth* of the formulas.
- In propositional logic, e.g., a formula is classically provable if and only if it is valid in every boolean model. So the truth value of a formula can be determined by inspecting a *truth table*. A formula like $p \wedge q$ is true if and only if both p and q are true.
- However, with new modalities, we can no longer determine the truth value of a formula by a truth table.
- The truth value of p does not determine the truth value of $\Box p$.
- Semantics for modal logics can be defined by introducing *possible worlds* evolving over time.
- The truth in a model is to say *where* the sentence is true or false.

Possible worlds

- In modal semantics, a set W of possible worlds is given. A valuation V assigns truth values to each propositional variable in each world, so p in world w may differ from p in world v.



- Worlds: $W := \{w_0, w_1, \cdots, w_7\}$
- w_1 and w_2 are accessible from w_0, ...
- w_3 p
- w_5 q

Modal Semantics

Definition (Frame)

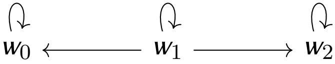
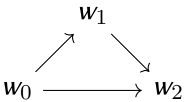
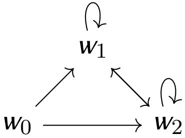
A *frame* is a pair (W, R) where:

- W is a set. The members of W are referred to as *worlds*/states. W is referred to as the *universe* of the frame.
- R is a binary relation on W. The relation R is known as the *accessibility relation*.

For a frame $\mathcal{F} := (W, R)$:

- If R is reflexive, then the frame $\mathcal{F}$ is called *reflexive*,
- If R is transitive, then the frame $\mathcal{F}$ is called *transitive*,
- and so on ...

Examples of different modal logics and their frame conditions

Logic	Frame Cond.	R shape example	
K	None	-	-
KD	Serial	$w_0 \to w_1 \to w_2 \to w_3 \circlearrowleft$	$\forall x \exists y R(x, y)$
KT	Reflexive		$\forall x (R(x, x))$
KB	Symmetrical	$w_0 \leftrightarrow w_1 \leftrightarrow w_2$	$\forall x \forall y (R(x, y) \to R(y, x))$
K4	Transitive		$\forall x \forall y \forall z (R(x, y) \land R(y, z) \to R(x, z))$
K5	Euclidean		$\forall x \forall y \forall z (R(x, y) \land R(x, z) \to R(y, z))$

Entailment

Definition (Model)

A *model* is a tuple $\mathcal{M} = (W, R, V)$ where (W, R) is a frame and V is a valuation function, $V : \Sigma \times W \to \{0, 1\}$, such that for all $w \in W$, $V(p, w) = 1$ if and only if p is true in w.

Definition (Entailment)

Let $\mathcal{M} = (W, R, V)$ be a model, and a world $w \in W$. The interpretation of a formula A in the world w is denoted by $w \models_{\mathcal{M}} A$. When the model is clear from the context, we write $w \models A$ instead of $w \models_{\mathcal{M}} A$. The entailment is defined inductively on A as follows:

$$
\begin{array}{ll}
w \models p & \text{if } \; V(p, w) = 1, \text{ for } p \in \Sigma. \\
w \models \top & \text{always true} \\
w \not\models \bot & \text{always false} \\
w \models \neg A & \text{iff } \; w \not\models A. \\
w \models (A \wedge B) & \text{iff } \; w \models A \; \text{ and } \; w \models B. \\
w \models (A \vee B) & \text{iff } \; w \models A \; \text{ or } \; w \models B.
\end{array}
$$

Definition (Entailment continued)

$$
\begin{array}{ll}
w \models (A \to B) & \text{iff } \ w \not\models A \ \text{ or } \ w \models B. \\
w \models \Box A & \text{iff } \ t \models A \text{ for all worlds } t \text{ such that } R(w,t). \\
w \models \Diamond A & \text{iff } \ t \models A \text{ for some world } t \text{ such that } R(w,t).
\end{array}
$$

- The relation $(\models)$ is called the *satisfaction*/forcing/evaluation relation and we can read $w \models A$ as:
  - w satisfies A, or
  - w forces A, or
  - A is true in w, or
  - w models A.
- Notice that $w \models A$ is uniquely determined by its value on propositional variables.
- Satisability: for a given $\mathcal{M}$ and a formula A, determine whether there is a world w in $\mathcal{M}$ such that $w \models A$.

Validity

Definition (Validity)

A formula A is *valid* in a model $\mathcal{M} := (W, R, V)$, denoted by $\mathcal{M} \models A$, if and only if $w \models_{\mathcal{M}} A$ for all $w \in W$.

Definition

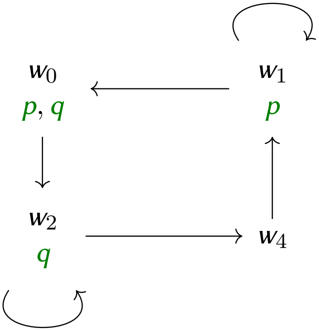
$\models A$ is valid if and only if for all models $\mathcal{M} := (W, R, V)$, $\mathcal{M} \models A$.

The following principles are valid:

- Necessitation: If $\models A$, then $\models \Box A$.
- Distribution: $\models \Box(A \to B) \to (\Box A \to \Box B)$.

Checking satisfiability

- $W = \{w_0, w_1, w_2, w_4\}$.
- $R = \{(w_0, w_1), (w_1, w_1), (w_2, w_1), (w_2, w_4), (w_4, w_1), (w_4, w_4)\}$.
- $\mathcal{M} := (W, R, V)$ where V is induced from the graph, e.g., $V(p, w_0) = 1$ and $V(q, w_4) = 0$.



1. $\mathcal{M} \models (p \wedge q) \rightarrow \Box\neg p$

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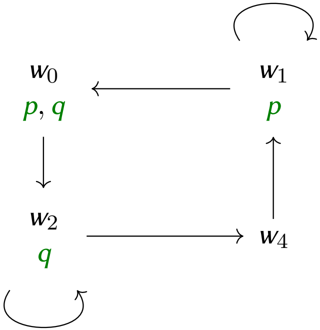

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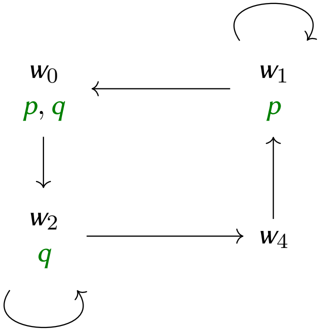

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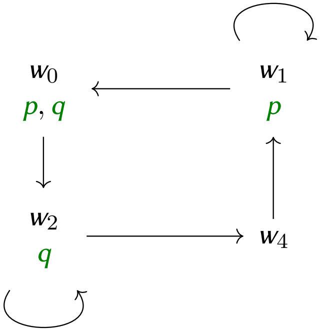

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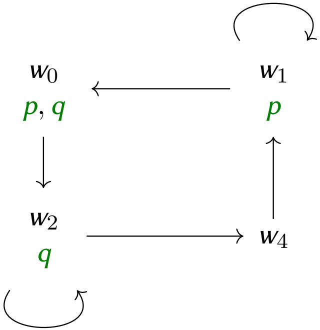

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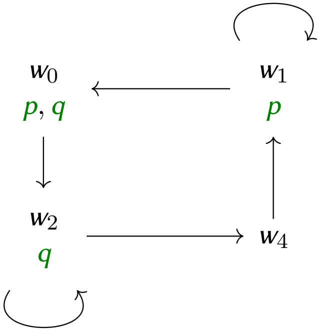

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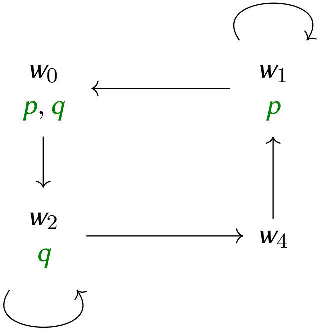

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Checking satisfiability

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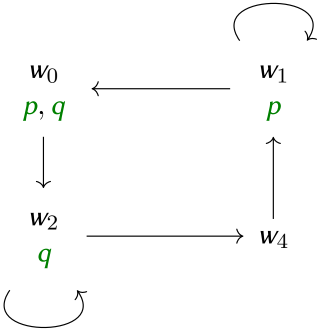

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8. $\mathcal{M} \models (\neg p \wedge q) \to \Diamond(\neg p \wedge q)$

Constructing models

Show that $A := \Diamond(p \lor q) \land \Box(\neg p)$ is Ref-satisfiable.

- Let's assume that A is true in a world w_1. We want to construct a model $\mathcal{M} = (W, R, V)$ such that R is reflexive.

Constructing models

Show that $A := \Diamond(p \vee q) \wedge \Box(\neg p)$ is Ref-satisfiable.

- Let's assume that A is true in a world w_1. We want to construct a model $\mathcal{M} = (W, R, V)$ such that R is reflexive.
- That means we need that $\Diamond(p \vee q)$ is true in w_1 and $\Box(\neg p)$ is true in w_1.

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- That means we need that $\Diamond(p \vee q)$ is true in w_1 and $\Box(\neg p)$ is true in w_1.
- 1. $w_1 \models \Diamond(p \vee q)$. This forces us to create a new world w_2 such that:
    - $R(w_1, w_2)$, and $R(w_i, w_i)$ for all $w_i \in W$.
    - $V(p \vee q, w_2) = 1$.
- 2. $w_1 \models \Box(\neg p)$.

Constructing models

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- 2. $w_1 \models \Box(\neg p)$. Then, for all worlds accessible from w_1, i.e., $R(w_1, w_i)$ for $w_i \in W$, we must have $V(\neg p, w_i) = 1$.

Constructing models

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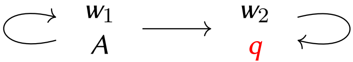
- That means we need that $\Diamond(p \vee q)$ is true in w_1 and $\Box(\neg p)$ is true in w_1.
- 1. $w_1 \models \Diamond(p \vee q)$. This forces us to create a new world w_2 such that:
  - $R(w_1, w_2)$, and $R(w_i, w_i)$ for all $w_i \in W$.
  - $V(p \vee q, w_2) = 1$.
- 2. $w_1 \models \Box(\neg p)$. Then, for all worlds accessible from w_1, i.e., $R(w_1, w_i)$ for $w_i \in W$, we must have $V(\neg p, w_i) = 1$. In this case, w_1 and w_2 are the only worlds accessible from w_1.
- Finally, because $p \vee q$ is true in w_2, q must be true in w_2.
- So, $V(p, w_1) = 0$, $V(q, w_1) = 0$, $V(p, w_2) = 0$, and $V(q, w_2) = 1$.

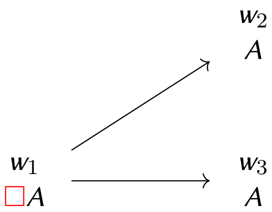


Rules to construct models

- There is at least a naive procedure to construct these models.
- As starting point, we have a world w_1 and a given formula.
- Notation $A1$ means that A is true and $A0$ means that A is false in the world.
- For the propositional fragment, we add the formula to the node of the world if it is true.
  - If $(A \wedge B)1$, then we *add* $A1$ and $B1$.
  - If $(A \wedge B)0$, then we *choose* $A0$ *or* $B0$.
  - If $(A \vee B)1$, then we *choose* $A1$ *or* $B1$.
  - If $(A \vee B)0$, then we *add* $A0$ and $B0$.
  - If $(A \rightarrow B)1$, then we *choose* $A0$ *or* $B1$.
  - If $(A \rightarrow B)0$, then we *add* $A1$ *and* $B0$.
  - If $(\neg A)0$, then we *add* $A1$.
  - If $(\neg A)1$, then we *choose* $A0$.
  - If $A \leftrightarrow B$ is true, then we *choose* $A1$ and $B1$ *or* $A0$ and $B0$.
  - If $A \leftrightarrow B$ is false, then we *choose* $A1$ and $B0$ *or* $A0$ and $B1$.
- For the modal fragment, . . .

Constructing models

- If $\Box A$ is true, then we *add* $A1$ in each of the accessible worlds.

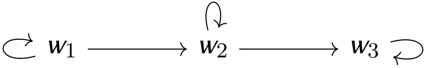


- If $\Box A$ is false, then we *create a new world* with $A0$ and *add arrows to the new world.*
- If $\Diamond A$ is true, then we *create a new world* with $A1$ and *add arrows to the new world.*
- If $\Diamond A$ is false, then we *add* $A0$ in each of the accessible worlds.

Example for constructing countermodels

Find a Ref-countermodel $\mathcal{M} := (W, R, V)$ for $\mathcal{M} \not\models_{\text{REF}} \Box p \to \Box\Box p$.

1. We assume that $V(\Box p, w_1) = 1$ and $V(\Box\Box p, w_1) = 0$, for some world w_1.
2. $R(w_1, w_1)$ because R is reflexive.
3. As $V(\Box p, w_1) = 1$, then $V(p, w_1) = 1$.
4. Since we have $V(\Box\Box p, w_1) = 0$, we create a new world w_2 where $V(\Box p, w_2) = 0$ and $R(w_1, w_2)$.
5. $R(w_2, w_2)$ because R is reflexive.
6. Because $V(p, w_1) = 1$, $V(p, w_2) = 1$ by def of $\Box$.
7. Because $V(\Box p, w_2) = 0$, we need to create a new world w_3 where $V(p, w_3) = 0$, $R(w_2, w_3)$, and $R(w_3, w_3)$.



Non-classical logics

- Intuitionistic logic:
  - Rejects the law of excluded middle ($\vdash A \vee \neg A$) and double negation elimination ($\vdash \neg\neg A \to A$)
  - Developed by Brouwer in early 1900s based on constructive principles
  - Initially lacked formal semantics, but now has several:
    - Kripke semantics (1965)
    - Beth semantics
    - Topological semantics
    - Algebraic semantics (Heyting algebras)
  - Can be embedded into classical modal logic via the Gödel-McKinsey-Tarski translation

Kripke structures

Definition (Kripke Frame)

A *Kripke frame* is a tuple $(W, \leq)$ where

- W is a set of worlds, and
- $\leq$ is a partial order on W.

- A world/state $w \in W$ represents a "state of knowledge."
- The relation $\leq$ is known as the *information order*.
- $w \leq t$ indicates that the world w has at least as much knowledge as the world t.
- Transitioning from w to t may involve gaining additional information.

Kripke structures

Definition (Kripke model)

A *Kripke model* is a tuple $(W, \leq, V)$ where

- $(W, \leq)$ is a Kripke frame, and
- V is a valuation function, $V : \Sigma \to \mathrm{Up}(W)$, such that:
  - $V(p) \subseteq W$ is the set of worlds where p is true.
  - $\mathrm{Up}(W)$ is defined as:

$$\mathrm{Up}(W) := \{S \subseteq W \mid \forall w \in S.\, \forall t \in W.\, w \leq t \implies t \in S\}.$$

- The set $\mathrm{Up}(W)$ is the set of (upper sets) all subsets of W that are closed under the information order $\leq$.
- What becomes true, it remains true as information increases.

The entailment relation

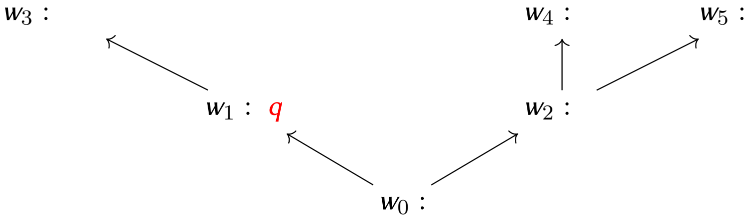
Definition (Entailment)

Let $\mathcal{M} := (W, \leq, V)$ be a Kripke model, and a world $w \in W$. The interpretation of a formula A in the world w is denoted by $w \models A$ and defined inductively on A as follows:

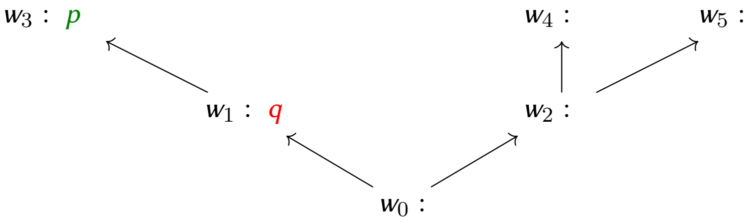
$w \models p$	iff $w \in V(p)$.
$w \models \top$	always.
$w \not\models \bot$	never.
$w \models \neg A$	iff, for every $t \in W$ such that $w \leq t$, then $t \not\models A$.
$w \models (A \wedge B)$	iff, $w \models A$ and $w \models B$.
$w \models (A \vee B)$	iff, $w \models A$ or $w \models B$.
$w \models (A \to B)$	iff, for every $t \in W$ such that $w \leq t$, and $t \models A$, then $t \models B$., i.e, if $t \in V(A)$, then $t \in V(B)$.

- Monotonicity: if $w \models_{\mathcal{M}} A$, then for all t such that $w \leq t$, then $t \models_{\mathcal{M}} A$.
- Kripke semantics is sound and complete with respect to the *intuitionistic* provability logic.

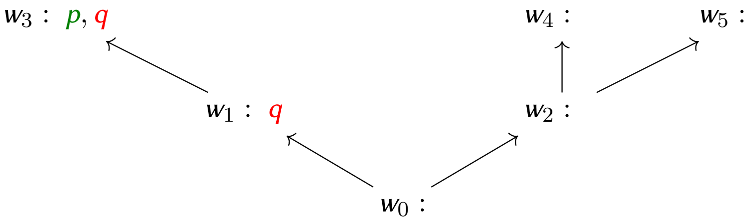
Example of a Kripke model



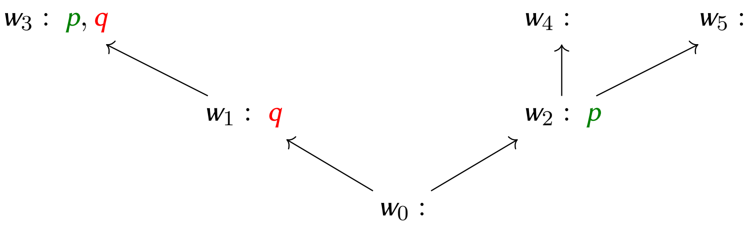
Example of a Kripke model



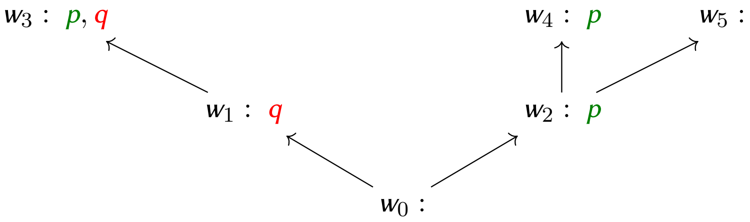
Example of a Kripke model



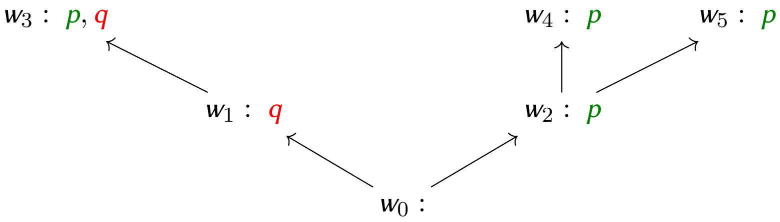
Example of a Kripke model



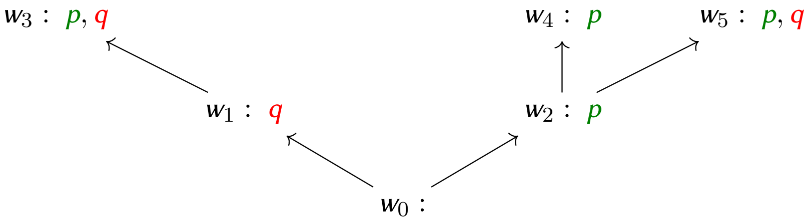
Example of a Kripke model



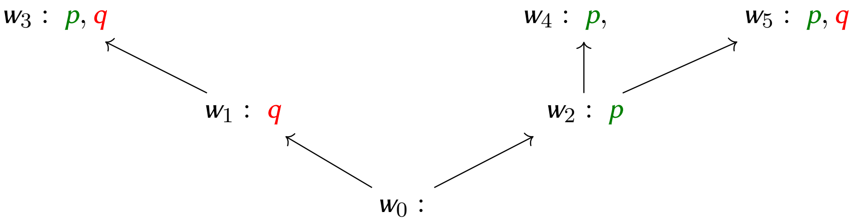
Example of a Kripke model



Example of a Kripke model

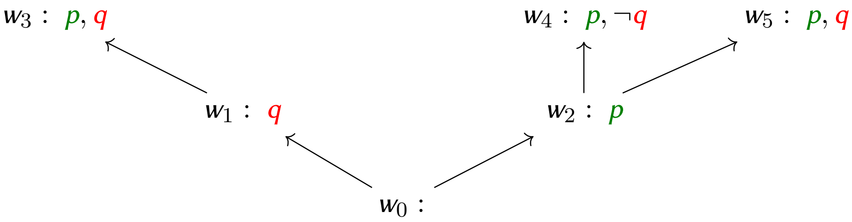


Example of a Kripke model



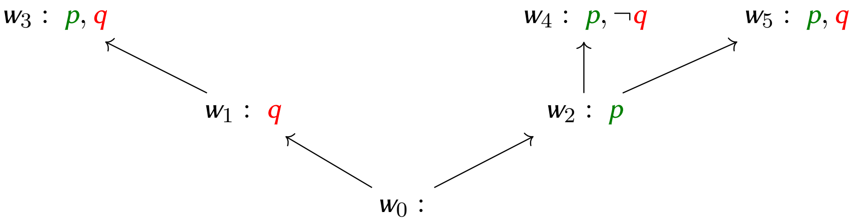
- Recall that $\neg A := A \to \bot$. To say that $\neg A$ is true at some world is to say that we never get A anywhere after.

Example of a Kripke model



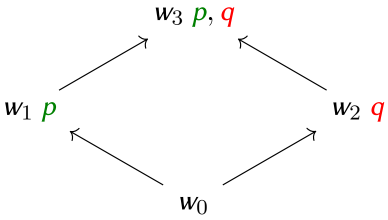
- Recall that $\neg A := A \to \bot$. To say that $\neg A$ is true at some world is to say that we never get A anywhere after.
- Notice that $\neg q$ is true in w_4, but neither q nor $\neg q$ can be true at w_0 since they conflict in different branches.

Example of a Kripke model



- Recall that $\neg A := A \to \bot$. To say that $\neg A$ is true at some world is to say that we never get A anywhere after.
- Notice that $\neg q$ is true in w_4, but neither q nor $\neg q$ can be true at w_0 since they conflict in different branches.
- LEM does not hold in this model, i.e., $A \vee \neg A$ is not valid in this model for any A. We don't have q or $\neg q$ at w_0. But also, p and $\neg p$ cannot be true at w_0.

Example of a Kripke model



No double negation elimination

1. The contrapositive of the soundness theorem says that if we can find a Kripke structure in which there is a world where a formula A is not satisfied, then A is not *intuitionistically provable.*
2. Let's show that $\neg\neg A \to A$ is not intuitionistically provable.
3. Consider the following Kripke structure.

$$w_0 \longrightarrow w_1 \ A$$

4. $w_0 \not\models A$.
5. We have that $w_0 \models \neg\neg A$. This is because there exists an extension of w_0 (namely, w_1) that does not force $\neg A$. We have that $w_1 \not\models \neg A$ because there is an extension of w_1 (namely, w_1 itself) that does force A.
6. At w_0, $\neg\neg A$ holds but A does not, proving $\neg(\neg\neg A \to A)$.

Bonus slides

Go back to classical modal logic

- We can translate intuitionistic logic into classical modal logic using the following translation $g : \mathcal{L}_{\text{int}} \to \mathcal{L}_{\text{KT4}}$. It is defined as follows by induction on the syntax of formulas:
  - $\top \mapsto \top$,
  - $\bot \mapsto \bot$,
  - $p \mapsto \Box p$,
  - $\neg A \mapsto \Box \neg A$,
  - $A \wedge B \mapsto A \wedge B$,
  - $A \vee B \mapsto A \vee B$, and
  - $A \to B \mapsto \Box(A \to B)$.
- We must add the following axioms to the normal modal logic (**K**):
  - T: $\Box A \to A$, and
  - 4: $\Box A \to \Box\Box A$.

Dealing with double negation

In terms of the forcing relation, we have the following:

- w forces $\neg A$ if and only if no extension of w forces A.
- Another way to say this is that further we go in the information order, we will eventually find a world that forces A.
- w does not force $\neg B$ if and only if some extension forces B.
- Another way to say this is that there is some information level at which B is forced. Recall that we never force $\bot$, and $\bot$ is locally equivalent to $B \wedge \neg B$.
- w forces $\neg\neg C$ if and only if no extension v forces $\neg C$, if and only if, for every extension v of w there is *an* extension t of v that forces C. We usually abbreviate this as: w forces $\neg\neg C$ if and only if the set of worlds that force C is **dense** *above* w.

It is also worth knowing that the forcing relation is often written $\Vdash$ instead of $\models$.